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Rational Conformal Field Theory and Multi-Wormhole Partition Function in 3-dimensional Gravity

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Abstract. We study the Turaev-Viro invariant as the Euclidean Chern-Simons-Witten gravity partition function with positive cosmological constant. After explaining why it can be identified as the partition function of 3-dimensional gravity, we show that the initial data of the TV invariant can be constructed from the duality data of a certain class of rational conformal field theories, and that, in particular, the original Turaev-Viro's initial data is associated with the A_{k+1} modular invariant WZW model. As a corollary we then show that the partition function $Z(M)$ is bounded from above by $Z((S^2 \times S^1)^{\sharp g}) = (S_{00})^{-2g+2} \sim \Lambda^{-\frac{3g-3}{2}}$, where g is the smallest genus of handlebodies with which M can be presented by Heegaard splitting. $Z(M)$ is generically very large near $\Lambda \sim +0$ if M is neither S^3 nor a lens space, and many-wormhole configurations dominate near $\Lambda \sim +0$ in the sense that $Z(M)$ generically tends to diverge faster as the “number of wormholes” g becomes larger.

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1 Introduction

Recently it has been shown that the Turaev-Viro (TV) topological invariant constructed as a state-sum over q -deformed spin-networks [1] provides us an interesting 3-dimensional Euclidean gravity model [2-5]. A crucial observation is that their construction strongly resembles the 3-dimensional Regge calculus studied by Ponzano and Regge [6] in the late 60's. In ref.[6] they assigned an $SU(2)$ $6j$ symbol to each simplex of a triangulated 3-manifold, and then summed them up over all spin-configurations with some edge and vertex factors, regarding spins as actual lengths of edges. They found that the summation can be written as a path-integral of 3-dimensional gravity action in the large spin region. Surprisingly, the TV invariant, constructed after more than twenty years, was completely in the same form as Ponzano-Regge's partition function, except that the quantities associated with representations of $SU(2)$ were replaced by those of q -deformed $SU(2)$ with a root of unity q . In fact the TV invariant is, as explained later, identified with the partition function of the Euclidean Chern-Simons-Witten (CSW) gravity theory with positive cosmological constant [7].¹ We first show that the TV invariant can be constructed from duality data of a certain class of rational conformal field theories (RCFT's), and that, in particular, the original Turaev-Viro's initial data which is relevant to 3-dimensional gravity is associated with the A_{k+1} modular invariant $SU(2)$ WZW model. As a corollary we then show that the partition function $Z(M)$ is bounded from above by $Z((S^2 \times S^1)^{\#g}) = (S_{00})^{-2g+2} \sim \Lambda^{-\frac{3g-3}{2}}$, where g is the smallest genus of handlebodies with which M can be presented by Hegaard splitting. This upper-bound argument is motivated by the work by Kohno [9], in which Witten's topological invariant (Jones polynomial as an expectation value of a Wilson line which carries identity representation)[10] is studied as a Hegaard splitting invariant, though in that context the estimation is for the lower-bound of g .

Several years ago a possible scenario that the wormhole sum may force the cosmological constant to be zero was proposed and discussed [11]. This idea has been also tested in $2+1$ -dimensional CSW gravity by summing up $S^2 \times S^1$ wormholes to obtain a rather negative result [8]. In this topological theory the partition function does not depend on size or location of wormholes, and the wormhole summation is reduced to a more tractable problem than in 4 dimensions. However, we still do not know a natural weight with which we sum over topologies since we do not know a 3-dimensional analogue of string field theory or matrix model, so it would be meaningful at this stage to study the

¹It should be noted that by CSW partition function we mean *pure* Chern-Simons partition function without factoring out the global diffeomorphisms [8]. See below.

cosmological constant dependence of a partition function with fixed topology. In general one can probe the large k behavior of the Chern-Simons (CS) partition function by the saddle-point approximation [8, 10, 12, 13, 14], in which a Gaussian integration around each extremum gives the Ray-Singer (RS) torsion multiplied by some phase factor. The full $SU(2) \times SU(2)$ CS partition function, which is considered as an Euclidean counterpart of the CSW gravity with positive cosmological constant (without exotic terms), is given by the absolute square of the sum (or the integration over the moduli) of such contributions. Instead, once one recognizes the correspondence between the TV invariant and the CS theory, one may use it to determine the exact cosmological constant behavior away from $\Lambda \sim (+)0$. We thus expect the TV invariant to be an alternative to the saddle-point approximation in 3-dimensional gravity.

This paper is organized as follows. In sect.2 we briefly review the construction of the TV invariant, and then explain how it is related to the CSW gravity theory. In sect.3 we introduce the axiom of RCFT [15] and show that the TV invariant can be constructed from a certain class of RCFT's. In sect.4 we study the TV invariant as multi-wormhole partition functions, and investigate their topology and cosmological constant dependence, in particular their behavior near $\Lambda \sim +0$. Sect.5 contains a brief summary of our results and a discussion. In appendix we give an elementary proof of the factorization formula of the TV invariant, which is used in sect.4.

2 3-dimensional gravity and the TV Invariant

2.1 Review of the TV invariant

We will begin by the definitions first. Let I be a finite set of “spin” variables. Assume that we have distinguished a set adm of unordered triples of elements of I . The triple (i, j, k) is said to be admissible if $(i, j, k) \in adm$. An ordered 6-tuple $(i, j, k, l, m, n) \in I^6$ is called admissible if the unordered (i, j, k) , (k, l, m) , (m, n, i) and (j, l, n) are admissible. Next assume that we are given a complex-valued function $\left| \begin{smallmatrix} i & j & k \\ l & m & n \end{smallmatrix} \right|$, which is called symbol, of admissible 6-tuple (i, j, k, l, m, n) . It is assumed to have the following symmetries:

$$\left| \begin{smallmatrix} i & j & k \\ l & m & n \end{smallmatrix} \right| = \left| \begin{smallmatrix} j & i & k \\ m & l & n \end{smallmatrix} \right| = \left| \begin{smallmatrix} i & k & j \\ l & n & m \end{smallmatrix} \right| = \left| \begin{smallmatrix} i & m & n \\ l & k & j \end{smallmatrix} \right| = \left| \begin{smallmatrix} l & m & k \\ i & j & n \end{smallmatrix} \right|, \quad (2.1)$$

so that it is naturally associated with a tetrahedron. A 6-tuple is admissible if and only if any of the four triples that forms a triangle of the associated tetrahedron is admissible.

Finally we assume that we are given a complex-valued function $w(i) \equiv w_i$ of I , and a complex number $w(\neq 0)$. The symbol $|\cdots|$, the functions w_i and the number w are collectively referred to as initial data.

By a colored tetrahedron we mean a tetrahedron with an element of I attached to each edge. By a coloring ϕ of edges $\{E_1, \dots, E_b\}$ we mean a mapping $\phi : \{E_1, \dots, E_b\} \rightarrow I$; in other word, the way how we assign spins to the edges.

After these definitions Turaev-Viro's theorem is stated as follows [1]. Let M be a compact triangulated 3-manifold. Let a be the number of vertices, e of which lie on the boundary ∂M . Let E_1, \dots, E_b be the edges of M , the first f of which belong to ∂M . Finally let T_1, \dots, T_d be the tetrahedra of M . Turaev and Viro introduced the following three conditions on the initial data [1]:

$$\begin{aligned}
(*) \quad & \sum_j w_j^2 w_{j_4}^2 \begin{vmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{vmatrix} \begin{vmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{vmatrix} = \delta_{j_4, j_6} \\
(**) \quad & \sum_j w_j^2 \begin{vmatrix} j_2 & a & j \\ j_1 & c & b \end{vmatrix} \begin{vmatrix} j_3 & j & e \\ j_1 & f & c \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ a & e & j \end{vmatrix} \\
& = \begin{vmatrix} j_{23} & a & e \\ j_1 & f & b \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{vmatrix} \\
(***) \quad & w^2 = w_j^{-2} \sum_{\substack{k, l \\ (j, k, l) \in adm}} w_k^2 w_l^2 \text{ for all } j \in I.
\end{aligned} \tag{2.2}$$

Here the sum in $(*)$ and $(**)$ are carried out over j such that all the symbols involved in them are defined. If the initial data satisfy these three conditions $(*)$, $(**)$ and $(***)$, then the quantity $\Omega_M(\alpha)$ such that

$$\Omega_M(\alpha) = \sum_{\phi \in adm(M, \alpha)} |M|_\phi \tag{2.3}$$

$$|M|_\phi = w^{-2a+e} \prod_{r=1}^f w_{\phi(E_r)} \prod_{s=f+1}^b w_{\phi(E_s)}^2 \prod_{t=1}^d |T_t^\phi| \tag{2.4}$$

is independent of the triangulation of M , but depends only on the topology of M and the coloring of ∂M . Here $adm(M, \alpha)$ is the set of all admissible colorings with the fixed coloring α of ∂M , and $|T_t^\phi|$ stands for the symbol associated to the t th tetrahedron.

The outline of the proof is the following. Due to the Alexander's theorem any two triangulated 3-manifolds with the same topology can be transformed each other by some sequence of the Alexander moves and the inverse transformations. If one passes from the triangulation to the dual cell subdivision (fig.1), an Alexander move can be generated

by compositions of three kinds of elementary moves, *i.e.* the bubble move \mathcal{B} , the lune move \mathcal{L} and the Matveev move \mathcal{M} (fig.2). They first translated Ω_M in terms of the dual cell subdivision, and then showed that it is invariant under \mathcal{L} , \mathcal{M} and \mathcal{B} if the initial data satisfy the condition $(*)$, $(**)$ and $(***)$, respectively.

For future convenience we define [16]

$$c(l, k) = w_k^{-2} w_l^{-2} \sum_j \delta(l, j, k) w_j^2 \quad (2.5)$$

and

$$\tilde{w}^2 = \sum_{i \in I} w_i^4, \quad (2.6)$$

where $\delta(l, j, k) = 1$ if (l, j, k) is admissible, 0 otherwise. Then

$$\begin{aligned} \sum_{i,j} w_i^2 w_j^2 \left| \begin{array}{ccc} i & j & k \\ l & m & k \end{array} \right|^2 &= \sum_j w_i^{-2} w_j^{-2} \delta(l, m, k) \delta(l, j, k) \\ &= w_k^2 c(l, k) \delta(l, m, k), \end{aligned} \quad (2.7)$$

or another calculation gives

$$= w_k^2 c(m, k) \delta(l, m, k), \quad (2.8)$$

so $c(l, k) = c(m, k)$ if (l, m, k) is admissible. I is said to be irreducible if for any $j, m \in I$ there exists a sequence l_1, \dots, l_n with $l_1 = j$, $l_n = m$ such that (l_i, l_{i+1}, l_{i+2}) is admissible for any $i = 1, \dots, n-2$. Obviously $c(i, j)$ is a constant if I is irreducible [1, 16]. In that case

$$\begin{aligned} (w^2) &= w_j^{-2} \sum_{k,l} w_k^2 w_l^2 \delta(j, k, l) \\ &= \sum_k w_k^4 \cdot w_j^{-2} w_k^{-2} \sum_l w_l^2 \delta(j, k, l) \\ &= \tilde{w}^2 c \quad (c = c(j, k)), \end{aligned} \quad (2.9)$$

and so $(***)$ is satisfied. This fact will be used in the next section.

2.2 Identification with the CSW gravity partition function

For the following reason $\Omega_M(\alpha)$ can be identified as the partition function of 3-dimensional quantum gravity [2]. Let us evaluate $\Omega_M(\alpha)$ on a tetrahedron as a triangulated 3-ball B^3 :

$$\Omega_{B^3}(\alpha) = w^{-2a+4} \prod_{r=1}^6 w_{\alpha(E_r)} \sum_{\phi(\alpha)} \prod_{s=7}^b w_s^2 \prod_{t=1}^d |T_t^\phi|. \quad (2.10)$$

Here α represents a fixed coloring of the boundary. We fix a root of unity of degree $2(k+2)$, q , such that q^2 is a primitive root of unity of degree $k+2$. As initial data we take²

$$\begin{aligned} I &= \{0, \frac{1}{2}, 1, \dots, \frac{k}{2}\} \quad (k \in \mathbf{Z}) \\ |T_i^\phi| &= (-1)^{\sum_{l=1}^6 j_l^{(i)}} \begin{Bmatrix} j_1^{(i)} & j_2^{(i)} & j_3^{(i)} \\ j_4^{(i)} & j_5^{(i)} & j_6^{(i)} \end{Bmatrix} \\ w_j^2 &= (-1)^{2j} [2j+1], \quad w^2 = -\frac{2(k+2)}{(q-q^{-1})^2}, \end{aligned} \quad (2.11)$$

where $\{\dots\}$ stands for the Racah-Wigner $\mathcal{U}_{q\text{su}(2)}$ q -6j symbol[17]:

$$\begin{aligned} & \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \\ &= \Delta(j_1, j_1, j_{12}) \Delta(j_3, j, j_{12}) \Delta(j_1, j, j_{23}) \Delta(j_3, j_2, j_{23}) \sum_{z \geq 0} (-1)^z [z+1]! \\ & \cdot ([z-j_1-j_2-j_{12}]! [z-j_3-j-j_{12}]! [z-j_1-j-j_{23}]! [z-j_3-j_2-j_{23}]! \\ & \cdot [j_1+j_2+j_3+j-z]! [j_1+j_3+j_{12}+j_{23}-z]! [j_2+j+j_{12}+j_{23}-z]!)^{-1} \end{aligned} \quad (2.12)$$

with

$$\Delta(a, b, c) = \sqrt{\frac{[-a+b+c]! [a-b+c]! [a+b-c]!}{[a+b+c+1]!}} \quad (2.13)$$

and $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. A triple $(i, j, l) \in I^3$ is defined to be admissible if $i+j+l \in \mathbf{Z}$, $\leq k+2$ and $i \leq j+l$, $j \leq l+i$, $l \leq i+j$.

Suppose that k is very large. Then up to $O(k^{-2})$ a q -6j symbol above becomes an ordinary Racah-Wigner 6j symbol of $\text{su}(2)$. In the semi-classical continuum limit [6, 18] (values of spin $\phi(E_s) \rightarrow \infty$, number of vertices $a \rightarrow \infty$) a 6j symbol behaves as [19]

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \approx \left(\frac{1}{12\pi V}\right)^{\frac{1}{2}} \cos\left(\sum_{i=1}^6 \theta_i J_i + \frac{\pi}{4}\right) \quad (2.14)$$

in the domain where J_i are uniformly large ($J = j + \frac{1}{2}$). Here V is the volume of the tetrahedron, and θ_i is the angle between the outer normal of the two faces which have the edge j_i in common, regarding J as actual length. Replacing q -6j symbols in (2.10) by (2.14), we obtain

² q_0 in ref.[1] corresponds to q we use in this paper.

$$\begin{aligned}
\Omega_M(\alpha) &= \left(\frac{k^3}{2\pi}\right)^{-2a+4} \prod_{r=1}^6 w_{\alpha(E_r)} \\
&\cdot \sum_{\phi} \prod_{s=7}^b (-1)^{2\phi(E_s)} (2\phi(E_s) + 1) \prod_{t=1}^d (-1)^{\sum_{l=1}^6 j_l^{(t)}} \begin{Bmatrix} j_1^{(t)} & j_2^{(t)} & j_3^{(t)} \\ j_4^{(t)} & j_5^{(t)} & j_6^{(t)} \end{Bmatrix} \\
&\approx \left(\frac{k^3}{2\pi}\right)^{-2a+4} \prod_{r=1}^6 w_{\alpha(E_r)} \sum_{\phi} \prod_{s=7}^b (-1)^{2\phi(E_s)} (2\phi(E_s) + 1) \\
&\cdot \prod_{t=1}^d (-1)^{\sum_{l=1}^6 j_l^{(t)}} \left(\frac{1}{12\pi V^{(t)}}\right)^{\frac{1}{2}} \cos\left(\sum_{l=1}^6 \theta_l^{(t)} (j_l^{(t)} + \frac{1}{2}) + \frac{\pi}{4}\right), \tag{2.15}
\end{aligned}$$

where $j_l^{(t)} = \phi(E_s)$ if the l th edge of the t th tetrahedron is E_s . If we take *only the positive frequency part* of cosine, we have

$$\begin{aligned}
&\Omega_M(\alpha) \text{ (positive frequency part)} \\
&\approx \left(\frac{k^3}{2\pi}\right)^{-2a+4} \left(\frac{i}{48\pi}\right)^{\frac{d}{2}} \prod_{r=1}^6 w_{\alpha(E_r)} \\
&\cdot \sum_{\phi} \prod_{s=7}^b (-1)^{2\phi(E_s)} (2\phi(E_s) + 1) \prod_{t=1}^d \frac{1}{\sqrt{V^{(t)}}} \exp\left\{i \sum_{l=1}^6 (\pi - \theta_l^{(t)}) (j_l^{(t)} + \frac{1}{2})\right\} \\
&= \left(\frac{k^3}{2\pi}\right)^{-2a+4} \left(\frac{i}{48\pi}\right)^{\frac{d}{2}} \left(\prod_{r=1}^6 w_{\alpha(E_r)}\right)^{-1} \\
&\cdot \sum_{\phi} \prod_{s=1}^b \left[\frac{(-1)^{2\phi(E_s)} (2\phi(E_s) + 1)}{(\prod_{j=1}^{n_s} V^{(s,j)})^{\frac{1}{12}}} \exp\left\{i \sum_{j=1}^{n_s} (\pi - \theta^{(s,j)}) (\phi(E_s) + \frac{1}{2})\right\} \right]. \tag{2.16}
\end{aligned}$$

In the last line we changed the summation over all tetrahedra to the double summation; first over n_s tetrahedra which have the edge E_s in common, and then over all edges.

If the triangulation ϕ were such that the manifold could be embedded in a flat Euclidean geometry, the sum $\sum_{j=1}^{n_s} (\pi - \theta^{(s,j)})$ should be 2π . In our case, however, ϕ -summation is carried out over colorings that generically can not be embedded in a flat geometry, so we may regard ϕ -summation as integration over metric. More precisely, the summation $\sum_{s=1}^b \sum_{j=1}^{n_s} (\pi - \theta^{(s,j)}) (\phi(E_s) - \frac{1}{2})$ can be considered as a realization of the Einstein-Hilbert action on a 3-dimensional simplicial decomposition [6]. To show this, suppose that we parallel-transport some tangent vector along a small loop. If the loop encloses no edges, the vector does not change after the parallel-transport. If, however, the loop encloses an edge E_s , the vector rotates by angle $\sum_{j=1}^{n_s} (\pi - \theta^{(s,j)}) - 2\pi$. Hence the curvature tensor has its support only at each edge. Integrating the scalar curvature

R over the interior of a thin cylinder C along E_s , we obtain

$$\begin{aligned}\int_C d^3x R &= (\phi(E_s) + \frac{1}{2}) \cdot \int_{\text{section of } C} d^2x R \\ &= (\sum_{j=1}^{n_s} (\pi - \theta^{(s,j)}) - 2\pi)(\phi(E_s) + \frac{1}{2}).\end{aligned}\quad (2.17)$$

Substituting (2.17) into (2.16) gives

$$(\text{const. which depends only on the boundary}) \times \int \mathcal{D}\phi \exp i \int_M \sqrt{g} R, \quad (2.18)$$

where

$$\mathcal{D}\phi = \sum_{\phi} \prod_{s=1}^b \frac{2\phi(E_s) + 1}{(\prod_{j=1}^{n_s} V^{(s,j)})^{\frac{1}{12}}}. \quad (2.19)$$

Therefore the positive frequency part of $\Omega_M(\alpha)$ can be seen as a partition function of 3-dimensional gravity with measure $\mathcal{D}\phi$ in the large k and the semi-classical continuum limit. This interesting suggestion on the relation between spin net-works and quantum gravity was made by Ponzano and Regge in the late 60's [6]. In fact, since they deal with classical $\text{su}(2)$ symbols, the expression for the partition function diverges. On the other hand, due to the restriction for the spin variables in q -6j symbols [17] the TV invariant is finite and well-defined. Recently the next-leading term in the action has been estimated and found to be a cosmological term with cosmological constant $\Lambda = \frac{4\pi^2}{k^2} + O(k^{-4})$ in this approximation [2].

Although it is interesting, some subtleties had been remaining unsolved until recently. First, we performed the summation only for the positive frequency part of cosine, but obviously other $2^d - 1$ terms do contribute to $\Omega_M(\alpha)$ and can not be ignored. Secondly, we are considering an Euclidean space-time manifold M , so it is strange that i appears in front of the action in the final form (2.16).

We may understand these points in the following way. The TV invariant is a triangulation independent topological invariant. If we regard it as a partition function, it must be the one of some topological field theory, while the Einstein gravity is not. But in 3-dimensions we have known for some time a topological gravity model: the Chern-Simons-Witten (CSW) gravity. Recall that the $\text{SU}(2) \times \text{SU}(2)$ Chern-Simons (CS) theory is on-shell equivalent to 3-dimensional Einstein gravity with positive cosmological constant. Everything goes well and is consistent if we regard $\Omega_M(\alpha)$ as a partition function of the CSW theory, rather than one of the Einstein gravity. Indeed, the CS theory has i factor in front of the action in the path-integral, and, moreover, since the CS gravity is a first order formalism, path-integration includes the sum over the orientation of

space-time:

$$\mathcal{D}e\mathcal{D}\omega e^i \int e \wedge R = \mathcal{D}|e|\mathcal{D}\omega \cdot 2 \cos \int |e| \wedge R. \quad (2.20)$$

So the appearance of cosine is naturally acceptable as the state-sum over parity (Note that our measure $\mathcal{D}\phi$ (2.19) is positive definite.).

On the other hand, by comparing the representation of the modular group induced from the Jones polynomial with the one from the TV invariant, it has been conjectured in [1] that for any closed oriented 3-manifold M the TV invariant Ω_M is equal to the absolute square of Witten's invariant [10] with corresponding level k . This relation can be checked in some simple topologies [3, 16, 20] as is checked later. Besides, as shown in appendix, the TV invariant satisfies the factorization formula characteristic to CS theories, and hence the check extends to arbitrary number of connected sums of such topologies.³ So we may write Ω_M as a partition function of the CS theory with two independent $SU(2)$ gauge group:

$$\Omega_M = \int \mathcal{D}A^+ \mathcal{D}A^- e^{i(S_{CS}[A^+] - S_{CS}[A^-])} \quad (2.21)$$

$$S_{CS}[A] = \int \epsilon^{ijk} (2A_i^a \partial_j A_k^a + \frac{2}{3} \epsilon_{abc} A_i^a A_j^b A_k^c). \quad (2.22)$$

This is the CSW Euclidean gravity partition function with positive cosmological constant. It agrees with the next-leading estimation for the asymptotic behavior of q -6j symbols [2], and the study of the physical Hilbert space of $ISO(3)$ CS theory [3]. Note that the relative sign between $S_{CS}[A^+]$ and $S_{CS}[A^-]$ comes from the fact that Ω_M is the absolute value square, and hence the "exotic" term is absent in the CSW action as it is in (2.18). We thus identify the TV invariant with an exact Euclidean CSW partition function and proceed further to see its relation to RCFT's.

3 TV Invariant from RCFT's

3.1 Axioms of RCFT

We will now explain how we can construct the TV invariant from RCFT. First, let us consider the dual cell subdivision of a tetrahedron. The boundary of the dual surfaces which lies on a 2-simplex of the tetrahedron forms a 3-point vertex, and hence the tetrahedron may be regarded as a matrix element which connects s - and t -channel amplitude (fig.3). Indeed, this is the definitions of 6j symbol, where a vertex represents a

³Turaev has announced that this fact has been proved [21].

composition of two representations. Thus, in particular, one may consider a tetrahedron as a fusion matrix of conformal blocks [15]. One of the most important properties of 2-dimensional conformal field theory is duality. In a RCFT the space of physical conformal blocks is finite-dimensional, so that any two N -point blocks with the same external lines are related by a sequence of fusion and braiding represented by some finite-dimensional matrices. Such a sequence is not unique in general, but the duality matrix can not depend on them (after the phase which comes from the framing is specified). Therefore the fusion and braiding matrices are required to satisfy some polynomial equations. This is the idea of ref.[15] of an axiomatic approach to RCFT. Representing a sequence of fusion by a triangulated 3-manifold M , where the boundaries of dual surfaces lying on F_{\pm} such that $\partial M = F_+ \cup F_-$ and $F_+ \cap F_- = \phi$ represents two conformal blocks, the fact that the duality matrix does not depend on the way of fusion implies that it does not depend on the triangulation of M . Thus we may expect the duality matrix of RCFT to be a 3-dimensional topological invariant. We will show that this is the case.

We will now summarize the axiom of RCFT [15]. Let I be a finite index set, each of which represents a primary field of the chiral algebra \mathcal{A} , with a distinguished element 0 that represents the identity operator. i^{\vee} is assumed to be the only field that produces the identity operator $\mathbf{1}$ by fusion with i . Let V_{jk}^i ($i, j, k \in I$) be a space of chiral vertex operator $\mathcal{H}_j \otimes \mathcal{H}_k \rightarrow \mathcal{H}_i$, where \mathcal{H}_i denotes the representation space of \mathcal{A} . A chiral vertex operator is an intertwiner of representations of \mathcal{A} , *i.e.* an operator such that commute with the action of \mathcal{A} . V_{jk}^i is assumed to be a finite dimensional: $\dim V_{jk}^i = N_{jk}^i$. We represent an element of V_{jk}^i by a trivalent vertex as usual:

$$(3.1)$$

We restrict ourselves to the case such that N_{jk}^i is either 0 or 1, so we can take the matrix representation for duality transformations. The genus 0 duality transformations are generated by the fusion F , the braiding $B(\pm)$ and the braiding on a single chiral vertex $\Omega(\pm)$ (not to be confused with Ω_M in the previous section), which are defined in the following picture:

$$= \sum_q F_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \quad (3.2)$$

$$= \sum_q B_{pq} \begin{bmatrix} i & j \\ k & l \end{bmatrix} (+) \quad (3.3)$$

$$= \Omega_{jk}^i (+) \quad (3.4)$$

and analogous relations for $B(-)$ and $\Omega(-)$. Then the genus 0 equations are

$$\sum_s F_{p_2 s} \begin{bmatrix} j & k \\ p_1 & b \end{bmatrix} F_{p_1 l} \begin{bmatrix} i & s \\ a & b \end{bmatrix} F_{sr} \begin{bmatrix} i & j \\ l & k \end{bmatrix} = F_{p_1 r} \begin{bmatrix} i & j \\ a & p_2 \end{bmatrix} F_{p_2 l} \begin{bmatrix} r & k \\ a & b \end{bmatrix} \quad (3.5)$$

$$\Omega_{lk}^m(\epsilon) F_{mn} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \Omega_{jk}^n(\epsilon) = \sum_r F_{mr} \begin{bmatrix} j & l \\ i & k \end{bmatrix} \Omega_{kr}^i(\epsilon) F_{rn} \begin{bmatrix} k & j \\ i & l \end{bmatrix} \quad (\epsilon = \pm). \quad (3.6)$$

The braiding matrix B is not independent but is written in terms of F and Ω as

$$B(\epsilon) = (\Omega(-\epsilon) \otimes \mathbf{1}) F (\mathbf{1} \otimes \Omega(\epsilon)). \quad (3.7)$$

They satisfy

$$B(+)B(-) = B(-)B(+) = \mathbf{1}. \quad (3.8)$$

For modular covariance we need three more constraints on the modular $S(j)$ and T matrices of the one-point function on the torus:

$$S(j)^2 = \pm C e^{i\pi\Delta_j} \quad (3.9)$$

$$(S(j)T)^3 = S(j)^2 \quad (3.10)$$

$$(S \otimes \mathbf{1}) F (\mathbf{1} \otimes \Theta(-) \Theta(+)) F^{-1} (S^{-1} \otimes \mathbf{1}) = F P F^{-1} (\mathbf{1} \otimes \Omega(-)), \quad (3.11)$$

where C is the conjugation operator, P is a flip operator which interchanges two chiral vertex operators, and $\Theta(\pm)$ is defined by

$$\Theta(+)\left(\begin{array}{c} \\ \end{array}\right) = e^{+i\pi(\Delta_k - \Delta_i - \Delta_j)} \quad (3.12)$$

and an analogous relation for $\Theta(-)$. Moore and Seiberg proved that once these conditions are satisfied, then all the constraints that may arise from the requirement of duality and modular covariance at higher genus are guaranteed by them [15]. They also showed that these general conditions that every RCFT must enjoy are enough for the proof of Verlinde's conjecture [22].

We would now like to show that the TV invariant in the last section can be indeed constructed from these duality data. First, the following relations can be easily checked:

$$\begin{aligned} F_{pr} \begin{bmatrix} j & k \\ i & l \end{bmatrix} &= \sigma_{13} \otimes \sigma_{23} F_{p^\vee r} \begin{bmatrix} k & j \\ l^\vee & i^\vee \end{bmatrix} \sigma_{13} \otimes \sigma_{13} P \\ &= \sigma_{12} \otimes \sigma_{12} P F_{pr^\vee} \begin{bmatrix} i^\vee & l \\ j^\vee & k \end{bmatrix} \sigma_{12} \otimes \sigma_{23} \\ &= \sigma_{123} \otimes \sigma_{132} P F_{p^\vee r^\vee} \begin{bmatrix} l & i^\vee \\ k^\vee & j \end{bmatrix} P \sigma_{123} \otimes \sigma_{132}, \end{aligned} \quad (3.13)$$

where for instance, $\sigma_{23} : V_{jk}^i \rightarrow V_{kj}^i$ such that

$$\sigma_{23}(V_{jk}^i)(\alpha \otimes \gamma \otimes \beta) = V_{kj}^i(\alpha \otimes \beta \otimes \gamma) \quad (\alpha \in \mathcal{H}_{i^\vee}, \beta \in \mathcal{H}_j, \gamma \in \mathcal{H}_k), \quad (3.14)$$

regarding V_{jk}^i as a function on $\mathcal{H}_{i^\vee} \otimes \mathcal{H}_j \otimes \mathcal{H}_k$. We would like to identify F as symbol of Turaev-Viro's initial data, so we will consider a self-conjugate RCFT, *i.e.* a theory in which i is identical to i^\vee for all fields $i \in I$. Besides, if all the eigenvalues of σ are $+1$, we can take the orbit of $\sigma(V_{jk}^i)$ as basis of $(V_{jk}^i, V_{kj}^i, V_{ki}^j, V_{ik}^j, V_{ij}^k, V_{ji}^k)$ so that

$$F_{pr} \begin{bmatrix} j & k \\ i & l \end{bmatrix} = F_{pr} \begin{bmatrix} k & j \\ l & i \end{bmatrix} = F_{pr} \begin{bmatrix} i & l \\ j & k \end{bmatrix} = F_{pr} \begin{bmatrix} l & i \\ k & j \end{bmatrix}. \quad (3.15)$$

If some eigenvalues of σ are -1 , (3.15) holds only up to signs on such basis in general. In the following we will consider for simplicity a class of theories in which all the eigenvalues of σ are $+1$. Such theories include the Virasoro minimal series. We will soon comment on some other cases in which some eigenvalues are -1 .

Now, in addition to (3.15) one can prove

$$F_{nk} \begin{bmatrix} i & j \\ l & p \end{bmatrix} F_{p0} \begin{bmatrix} k & k \\ l & l \end{bmatrix} = F_{pi} \begin{bmatrix} j & k \\ n & l \end{bmatrix} F_{n0} \begin{bmatrix} i & i \\ l & l \end{bmatrix} \quad (3.16)$$

from the pentagon identity (3.5). Normalizing the 0th column of F as

$$F_{k0} \begin{bmatrix} i & i \\ j & j \end{bmatrix} = \sqrt{\frac{F_i F_j}{F_k}} \quad (\text{"good gauge"}), \quad (3.17)$$

where $F_k = F_{00} \begin{bmatrix} k & k \\ k & k \end{bmatrix}$, (3.16) reads

$$\sqrt{F_n F_k F_{nk}} \begin{bmatrix} i & j \\ l & p \end{bmatrix} = \sqrt{F_p F_i F_{pi}} \begin{bmatrix} j & k \\ n & l \end{bmatrix}. \quad (3.18)$$

3.2 Construction of initial data

Due to (3.15) and (3.18) we may take

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{F_n F_k F_{nk}} \begin{bmatrix} i & j \\ l & m \end{bmatrix} \quad (3.19)$$

as symbol with full tetrahedral symmetry. Combining (3.19) and (3.5), $(**)$ of (2.2) is satisfied if we take $w_j^2 = F_j^{-1}$. Furthermore, combining (3.7) and (3.8) we find

$$F_{pq} \begin{bmatrix} j & l \\ i & k \end{bmatrix} F_{qr} \begin{bmatrix} j & k \\ i & l \end{bmatrix} = \delta_{pr}, \quad (3.20)$$

which implies the condition (*). Thus we have shown that we can construct the TV invariant from duality data of a self-conjugate RCFT with irreducible I . This is the case with the Virasoro minimal series (and the SU(2) WZW model), since successive fusions of the “shift” operators [23], like $\phi_{1,2}$ and $\phi_{2,1}$ operators in the former theory, connect any two primary fields in them.

In the SU(2) WZW model it is well-known that the duality matrices are written by using the $\mathcal{U}_q\text{su}(2)$ q -6j symbol, and in particular [24]

$$F_{kn} \begin{bmatrix} i & m \\ j & l \end{bmatrix} = \sqrt{[2k+1][2n+1]}(-1)^{i+j+l+m} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}. \quad (3.21)$$

An eigenvalue of σ can be ± 1 in this model. For example, the eigenvalue of σ_{23} on V_{jj}^i is $+(-)1$ if the representation i occurs (anti-)symmetrically in the tensor product $\mathcal{H}_j \otimes \mathcal{H}_j$. However, one can take appropriate basis of the chiral vertex operators (Racah-Wigner normalization) to obtain a symmetric symbol

$$\left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| = (-1)^{i+j+k+l+m+n} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}. \quad (3.22)$$

Hence the original Turaev-Viro’s initial data (2.11) is associated with the A_{k+1} modular invariant SU(2) WZW model. Although we have restricted ourselves to the simplest RCFT’s, we may do the same thing in other self-conjugate RCFT’s⁴ with left right factorized N_{jk}^i , such as the $D_{2\rho+2}$ with ρ even, E_6 and E_8 modular invariant SU(2) WZW model [26, 27]. The latter two are easy because N_{jk}^i is either 0 or 1 also in these models. We will list the initial data constructed from those:

• E_6 case

$$\begin{aligned} \text{fusion rule} \quad &: \quad \psi \times \psi = \mathbf{1} \\ &\quad \psi \times \sigma = \sigma \\ &\quad \sigma \times \sigma = \mathbf{1} + \psi \end{aligned}$$

$$\begin{aligned} \left| \begin{array}{ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array} \right| &= 1 & \left| \begin{array}{ccc} \mathbf{1} & \psi & \psi \\ \mathbf{1} & \psi & \psi \end{array} \right| &= 1 \\ \left| \begin{array}{ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \psi & \psi & \psi \end{array} \right| &= 1 & \left| \begin{array}{ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \sigma & \sigma & \sigma \end{array} \right| &= 2^{-\frac{1}{4}} \end{aligned}$$

⁴For non-self-conjugate theories Durhuus *et. al.* have constructed a generalized topological invariant to such cases [25], in which symbols possess only the half of the tetrahedral symmetry that preserves the orientation.

$$\begin{aligned}
\left| \begin{array}{ccc} \sigma & \sigma & \mathbf{1} \\ \psi & \psi & \sigma \end{array} \right| &= 2^{-\frac{1}{4}} & \left| \begin{array}{ccc} \sigma & \sigma & \mathbf{1} \\ \sigma & \sigma & \mathbf{1} \end{array} \right| &= 2^{-\frac{1}{2}} \\
\left| \begin{array}{ccc} \sigma & \sigma & \mathbf{1} \\ \sigma & \sigma & \psi \end{array} \right| &= 2^{-\frac{1}{2}} & \left| \begin{array}{ccc} \sigma & \sigma & \psi \\ \sigma & \sigma & \psi \end{array} \right| &= -2^{-\frac{1}{2}} \\
w_{\mathbf{1}}^2 &= 1 & w_{\sigma}^2 &= 2^{\frac{1}{2}} \\
w_{\psi}^2 &= 1 & w^2 = \tilde{w}^2 &= 4
\end{aligned} \tag{3.23}$$

$\cdot E_8$ case

fusion rule : $\varphi \times \varphi = \mathbf{1} + \varphi$

$$\begin{aligned}
\left| \begin{array}{ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array} \right| &= 1 & \left| \begin{array}{ccc} \mathbf{1} & \varphi & \varphi \\ \mathbf{1} & \varphi & \varphi \end{array} \right| &= [2]^{-1} \\
\left| \begin{array}{ccc} \mathbf{1} & \varphi & \varphi \\ \varphi & \mathbf{1} & \mathbf{1} \end{array} \right| &= [2]^{-1} & \left| \begin{array}{ccc} \mathbf{1} & \varphi & \varphi \\ \varphi & \varphi & \varphi \end{array} \right| &= -[2]^{-1} \\
\left| \begin{array}{ccc} \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi \end{array} \right| &= -[2]^{-2}
\end{aligned}$$

$$\begin{aligned}
w_{\mathbf{1}}^2 &= 1 & w_{\varphi}^2 &= [2] \\
w^2 = \tilde{w}^2 &= 1 + [2]^2 & [2] &= \frac{\sin \frac{2\pi}{5}}{\sin \frac{\pi}{5}},
\end{aligned} \tag{3.24}$$

where the primary fields are the ones of WZW models with extended chiral algebra, *i.e.* level-1 C_2 (E_6 case) and level-1 G_2 (E_8 case), respectively [28, 29]. The fusion rule for the E_6 case are the same as the Ising one, and the initial data (3.23) are easily read off from the appendix of ref.[15]. The fusion rule for the E_8 case are obtained from level-3 $SU(2)$ WZW model by restricting primary fields to integer-spin ones, and hence the initial data (3.24) are given by q -6j symbols.

It is useful to relate \tilde{w} with 00 entry of the modular S matrix. It is known that the condition (3.11) can be used to solve for S in terms of F and B , and in particular

$$S_{i0} = S_{00} F_i^{-1} \tag{3.25}$$

in this gauge. S_{00} can be fixed by unitarity of S :

$$S_{00}^2 = \tilde{w}^{-2}. \tag{3.26}$$

Note that

$$\begin{aligned}
1 &= \sum_l w_l^2 \left| \begin{array}{ccc} i & k & l \\ k & i & 0 \end{array} \right| \left| \begin{array}{ccc} k & k & 0 \\ i & i & l \end{array} \right| \\
&= \sum_l w_l^2 w_i^{-2} w_k^{-2} \delta(i, k, l) \\
&= c(i, k),
\end{aligned} \tag{3.27}$$

and hence

$$\tilde{w}^2 = w^2 = S_{00}^{-2}. \tag{3.28}$$

This relation is known in the case associated with the A_{k+1} modular invariant $SU(2)$ WZW model (the original Turaev-Viro's initial data). It can be easy to check that (3.28) holds true also for the E_6 and the E_8 case. We would like to stress here that it is a direct consequence of the requirement of modular covariance of RCFT.

3.3 Examples

We will now calculate the Turaev-Viro invariant for S^3 associated with duality data of a RCFT. It would be most easily done by presenting an S^3 as two 3-ball B^3 whose boundaries are glued together, but to illustrate the idea for finding upper bound in the next section we will present it here as two solid tori such that the boundary of the one is identified with that of the other after the modular S transformation.

Since the TV invariant does not depend on the triangulation within the manifold, we may take any triangulation of the solid torus. A convenient choice is such that its boundary, which is a torus, is triangulated as in fig.4 with opposite sides of the rectangle identified. Its dual graph represents a genus 3 conformal block. It is straightforward to see that for such coloring $\alpha = \alpha(j, l, i, k, i', k')$:

$$\Omega_{D^2 \times S^1}(\alpha(j, l, i, k, i', k')) = \frac{w_j w_l}{w^2} \delta_{ii'} \delta_{kk'}. \tag{3.29}$$

The modular S transformation on $\alpha(j, l, i, k, i', k')$ is performed by interchanging a - (meridian) and b - (longitude) cycle ($a \mapsto b$, $b \mapsto -a$), and it maps the genus 3 conformal block to another one. These two blocks are related by fusing twice (fig.5), and hence

$$\begin{aligned}
&\Omega_{D^2 \times S^1}(S(\alpha(j, l, i, k, i', k'))) \\
&= \sum_{j', l'} F_{ll'} \begin{bmatrix} i & k \\ i' & k' \end{bmatrix} F_{jj'} \begin{bmatrix} i & k \\ i' & k' \end{bmatrix} \Omega_{D^2 \times S^1}(\alpha(l', j', i, i', k, k')).
\end{aligned} \tag{3.30}$$

It may be noted that in the case of the Jones polynomial the modular transformation is represented by the modular S matrix, while in our case it is represented by the duality matrices. In contrast to the former case there is no phase ambiguity since it depends only on the coloring of the boundary [1]. It is also noticed that the F transformation induces the Alexander move on the triangulated 2-surface. Combining (3.29) and (3.30) for S^3

$$\begin{aligned}
\Omega_{S^3} &= \sum_{j,l,i,k,i',k'} \Omega_{D^2 \times S^1}(\alpha(j,l,i,k,i',k')) \Omega_{D^2 \times S^1}(S(\alpha(j,l,i,k,i',k'))) \\
&= \sum_{j,l,i,k,i',k'} \sum_{j',l'} F_{ll'} \begin{bmatrix} i & k \\ i' & k' \end{bmatrix} F_{jj'} \begin{bmatrix} i & k \\ i' & k' \end{bmatrix} \frac{w_{l'} w_{j'}}{w^2} \delta_{ik} \delta_{i'k'} \frac{w_j w_l}{w^2} \delta_{ii'} \delta_{kk'} \\
&= \sum_{j,l,i} \sum_{j',l'} F_{ll'} \begin{bmatrix} i & i \\ i & i \end{bmatrix} F_{jj'} \begin{bmatrix} i & i \\ i & i \end{bmatrix} \frac{w_{l'} w_{j'}}{w^2} \frac{w_j w_l}{w^2} \\
&= \sum_{j,l,i} \sum_{j',l'} F_{ll'} \begin{bmatrix} i & i \\ i & i \end{bmatrix} F_{l'0} \begin{bmatrix} i & i \\ i & i \end{bmatrix} F_{jj'} \begin{bmatrix} i & i \\ i & i \end{bmatrix} F_{j'0} \begin{bmatrix} i & i \\ i & i \end{bmatrix} \frac{w_i^4 w_j w_l}{w^4} \\
&= \sum_{j,l,i} \delta_{l0} \delta_{j0} \frac{w_i^4 w_j w_l}{w^4} \\
&= \frac{\tilde{w}^2}{w^4} \\
&= w^{-2} \\
&= S_{00}^2.
\end{aligned} \tag{3.31}$$

Here in the first line we do not need to reverse one of $\alpha(j,l,i,k,i',k')$ because the TV invariant is independent of the orientation of the manifold. This result is also well-known in the case of the original Turaev-Viro's initial data, and is consistent with the fact that the TV invariant is an absolute value square of Witten's invariant. For the same reason, the TV invariant associated with the E_6 and the E_8 modular invariant may be identified as the $SO(5) \times SO(5)$ and the $G_2 \times G_2$ CS partition function, respectively.

We can calculate also for $S^2 \times S^1$ in the same way and find

$$\begin{aligned}
\Omega_{S^2 \times S^1} &= \sum_{j,l,i,k,i',k'} (\Omega_{D^2 \times S^1}(\alpha(j,l,i,k,i',k')))^2 \\
&= 1.
\end{aligned} \tag{3.32}$$

Hence obviously Ω_{S^3} is always smaller than $\Omega_{S^2 \times S^1}$. In the next section we will generalize this fact to multi-wormhole partition functions.

4 Multi-wormhole partition function

We will now go back to the case of the original Turaev-Viro's initial data, which is relevant to the CSW gravity. It is known that any closed, orientable 3-manifold M is presented by so-called ‘‘Hegaard splitting’’ [30]. We say a 3-manifold M admits a Hegaard splitting of genus g if M is obtained by identifying boundaries of a pair of genus g handlebodies (M_1, M_2) through a modular transformation φ , *i.e.*

$$M = M_1 \cup_{\varphi} M_2, \quad \partial M_1 \sim -\partial M_2 \sim \Sigma_g. \quad (4.1)$$

Here Σ_g denotes a genus g 2-surface. ∂M_1 is identified with $\varphi(\partial M_2)$ after reversing the orientation. Regarding Ω_M as the CSW gravity partition function $Z(M)$, we can in principle calculate $Z(M)$ for any closed, orientable 3-manifold M .

We first consider a genus g Hegaard splitting of g times connected sum $(S^2 \times S^1) \# \dots \# (S^2 \times S^1)$ of $S^2 \times S^1$. Here a 3-manifold M is said to be a connected sum $M_1 \# M_2$ of M_1 and M_2 if M is obtained by cutting 3-balls from each of M_1 and M_2 , and then identifying the resulting boundaries. One can obtain $(S^2 \times S^1)^{\#g}$ by taking $\varphi = \text{identity}$ and gluing two genus g handlebodies together. So

$$\begin{aligned} Z((S^2 \times S^1)^{\#g}) &\equiv \Omega_{(S^2 \times S^1)^{\#g}} \\ &= \sum_{\alpha} (\Omega_{H_g}(\alpha))^2, \end{aligned} \quad (4.2)$$

where H_g denotes the handlebody of genus g , and α is a coloring of some triangulation of $\Sigma_g \sim \partial H_g$. On the other hand we can calculate $\Omega_{(S^2 \times S^1)^{\#g}}$ by using the factorization formula:

$$\frac{\Omega_{M_1 \# M_2}}{\Omega_{S^3}} = \frac{\Omega_{M_1}}{\Omega_{S^3}} \frac{\Omega_{M_2}}{\Omega_{S^3}}, \quad (4.3)$$

which we will prove in appendix.⁵ (4.3) is a characteristic property of the partition function of the CS theory [10]; the fact that the TV invariant satisfies (4.3) is a reflection of the equivalence to the CSW theory. Hence

$$\begin{aligned} Z((S^2 \times S^1)^{\#g}) &= \left(\frac{\Omega_{S^2 \times S^1}}{\Omega_{S^3}} \right)^g \cdot \Omega_{S^3} \\ &= (\Omega_{S^3})^{-g+1} \\ &= \left(-\frac{(q - q^{-1})^2}{2(k+2)} \right)^{-g+1}, \end{aligned} \quad (4.4)$$

⁵(4.3) has been proved also in ref.([16]) by modifying the construction of invariants for 3-manifolds with boundary.

where

$$\begin{aligned}\Omega_{S^3} &= S_{00}^2 \\ &= -\frac{(q - q^{-1})^2}{2(k+2)} \quad (q = e^{\frac{i\pi}{k+2}})\end{aligned}\tag{4.5}$$

has been used. Note that in the large k limit $Z(S^3) \equiv \Omega_{S^3}$ behaves as

$$Z(S^3) \sim k^{-3} \sim \Lambda^{\frac{3}{2}}.\tag{4.6}$$

Now, how about other topologies? Suppose that M admits a genus g Heegaard splitting with gluing transformation φ . We write

$$Z(M) = \sum_{\alpha} \Omega_{H_g}(\alpha) \Omega_{H_g}(\varphi(\alpha)).\tag{4.7}$$

In the previous section we saw that the modular transformation on the torus was described by the duality matrices between two genus 3 conformal blocks represented by the dual graphs of triangulated tori. In general the modular group is generated by the Dehn twists along non-contractible homology 1-cycles, under which a conformal block, represented by the dual graph of a triangulated boundary, obviously does not change its genus. Hence we know that these two conformal blocks are related by the duality matrices uniquely, at least up to phase which comes from the framing. But in our case there is no such phase ambiguity because the TV invariant is real except boundary factors, which are fixed. Thus the modular transformation is described by the duality matrices between the conformal blocks associated with the dual graphs. Since the duality matrices are unitary, we use the Cauchy-Schwartz inequality to obtain

$$Z(M) \leq Z((S^2 \times S^1)^{\#g}).\tag{4.8}$$

Hence the partition function $Z(M)$ is bounded by $Z((S^2 \times S^1)^{\#g})$, where g is the smallest genus in which M can be presented by Heegaard splitting. This is in agreement with the estimation for Witten's invariant by Kohno [9] together with the fact that the TV invariant is its absolute square. Since

$$Z((S^2 \times S^1)^{\#g}) \sim \Lambda^{-\frac{3g-3}{2}}\tag{4.9}$$

in the large k , this upper bound diverges as $\Lambda \rightarrow 0$ if $g \geq 2$. However, the ratio $\frac{Z(M)}{Z((S^2 \times S^1)^{\#g})}$ is generically $O(1)$ unless the “angle” between $\Omega_{H_g}(\alpha)$ and $\Omega_{H_g}(\varphi(\alpha))$ is accidentally very near $\frac{\pi}{2}$. Therefore in the CSW gravity we may say that $Z(M)$ is generically very large

if M is neither S^3 nor a lens space, and many-wormhole configurations dominate near $\Lambda \sim +0$ in the sense that $Z(M)$ generically tends to diverge faster as the “number of wormholes” g becomes larger. However, in summation over wormholes one should take their statistics into account, and hence $g + 1$ wormhole partition function is suppressed by $((g + 1)!)^{-1}$ if one assumes that wormholes have bosonic statistics. Besides, it has been argued [8] that one should consider the slide diffeomorphism of 3-manifolds as a part of gauge group, and then a contribution from a many-wormhole configuration would be more suppressed in the wormhole summation.

5 Conclusion

In this paper we have studied the TV invariant as the partition function of the Euclidean CSW gravity with positive cosmological constant. We have shown that initial data of the TV invariant can be constructed from the duality matrices of a self-conjugate RCFT with symmetrizable fusion matrix, and that in particular the original Turaev-Viro’s initial data is associated with those of the A_{k+1} modular invariant $SU(2)$ WZW model. The partition function $Z(M)$ has been shown to be bounded from above by $Z((S^2 \times S^1)^{\#g}) = (S_{00})^{-2g+2} \sim \Lambda^{-\frac{3g-3}{2}}$, where g is the smallest genus of handlebodies with which M can be presented by Heegaard splitting. $Z(M)$ is generically very large near $\Lambda \sim +0$ if M is neither S^3 nor a lens space, and many-wormhole configurations dominate near $\Lambda \sim +0$ in the sense that $Z(M)$ generically tends to diverge faster as the “number of wormholes” g becomes larger.

The fact that the value of the TV invariant on S^3 is always S_{00}^2 is a direct consequence of modular covariance of RCFT, though it is still obscure intrinsically why the topological invariant constructed from the duality matrices in such a way gives the absolute square of Witten type topological invariant. The TV invariant may be considered as a lattice realization of the CS partition theory, so it would be also interesting to ask whether it can be generalized to $2 + 1$ -dimensional gravity. In that case we may need to study the duality matrix of WZW theories with some non-compact gauge groups, in which we do not know any distinguished finite set of representations such as “good” representations in the $\mathcal{U}_{qsl}(2)$. In that sense it is not clear how to do so because rationality of underlying CFT is essential in regularization for the divergence in the state-sum on a lattice.

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Appendix

In this appendix we will give an elementary proof of the factorization formula (4.3):

$$\frac{\Omega_{M_1 \# M_2}}{\Omega_{S^3}} = \frac{\Omega_{M_1}}{\Omega_{S^3}} \frac{\Omega_{M_2}}{\Omega_{S^3}} \quad (4.3)$$

Consider first a triangulated cylinder $D^2 \times [0, 1]$ as shown in fig.6. Such coloring of the triangulation of its boundary is denoted by β . It is made of three tetrahedra, so

$$\begin{aligned} \Omega_{D^2 \times [0,1]}(\beta) &= \left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| \left| \begin{array}{ccc} p & m & i' \\ l & q & k \end{array} \right| \left| \begin{array}{ccc} i' & j' & k' \\ r & q & l \end{array} \right| \\ &\quad \cdot w_i w_j w_k w_{i'} w_{j'} w_{k'} w_l w_m w_n w_p w_q w_r w^{-6}. \end{aligned} \quad (A.1)$$

Gluing two such cylinders together, we have

$$\begin{aligned} &\Omega_{S^2 \times [0,1]}(\gamma(i, j, k), \gamma(i', j', k')) \\ &= \sum_{\beta} (\Omega_{D^2 \times [0,1]}(\beta))^2 (w_i w_j w_k w_{i'} w_{j'} w_{k'} w^{-6})^{-1} \\ &= \sum_{l,m,n,p,q,r} \left(\left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| \left| \begin{array}{ccc} p & m & i' \\ l & q & k \end{array} \right| \left| \begin{array}{ccc} i' & j' & k' \\ r & q & l \end{array} \right| w_l w_m w_n w_p w_q w_r \right)^2 \\ &\quad \cdot w_i w_j w_k w_{i'} w_{j'} w_{k'} w^{-6} \\ &= \sum_{l,m,q} \left(\sum_n \left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right|^2 w_k^2 w_n^2 \right) \left(\sum_p \left| \begin{array}{ccc} p & m & i' \\ l & q & k \end{array} \right|^2 w_k^2 w_n^2 \right) \left(\sum_r \left| \begin{array}{ccc} i' & j' & k' \\ r & q & l \end{array} \right|^2 w_k^2 w_n^2 \right) \\ &\quad \cdot w_i w_j w_k^{-1} w_{i'}^{-1} w_{j'} w_{k'} w_m^2 w_q^2 w^{-6} \\ &= \sum_{l,m,q} w_i w_j w_k^{-1} w_{i'}^{-1} w_{j'} w_{k'} w_m^2 w_q^2 w^{-6} \delta(i', q, l) \delta(l, m, k) \\ &= \sum_l w_i w_j w_k w_{i'} w_{j'} w_{k'} w_l^4 w^{-6} c^2 \\ &= w_i w_j w_k w_{i'} w_{j'} w_{k'} w^{-4} c \end{aligned} \quad (A.2)$$

where $\gamma(i, j, k)$ represents the coloring of the triangulation of S^2 (fig.7). On the other hand, we can calculate $\Omega_{B^3}(\gamma(i, j, k))$ by gluing two tetrahedra (fig.8) to obtain

$$\begin{aligned} \Omega_{B^3}(\gamma(i, j, k)) &= \sum_{l,m,n} \left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right|^2 w_i w_j w_k w_l^2 w_m^2 w_n^2 w^{-5} \\ &= w_i w_j w_k w^{-3}. \end{aligned} \quad (A.3)$$

Therefore

$$\Omega_{S^2 \times [0,1]}(\gamma(i, j, k), \gamma(i', j', k')) = \Omega_{B^3}(\gamma(i, j, k)) \Omega_{B^3}(\gamma(i', j', k')) \cdot w^2 c. \quad (\text{A.4})$$

Using $\Omega_{S^3} = (w^2 c)^{-1}$ (3.31), we have

$$\frac{\Omega_{S^2 \times [0,1]}(\gamma(i, j, k), \gamma(i', j', k'))}{\Omega_{S^3}} = \frac{\Omega_{B^3}(\gamma(i, j, k))}{\Omega_{S^3}} \frac{\Omega_{B^3}(\gamma(i', j', k'))}{\Omega_{S^3}}. \quad (\text{A.5})$$

(4.3) immediately follows from (A.5).

References

- [1] V. G. Turaev and O. Y. Viro, LOMI preprint (1990)
- [2] S. Mizoguchi and T. Tada, Phys. Rev. Lett. 68 (1992) 1795
- [3] H. Ooguri and N. Sasakura, Mod. Phys. Lett. A6 (1991) 3591
- [4] F. Archer and R. M. Williams, Phys. Lett. B273 (1991) 438
- [5] H. Ooguri, Kyoto RIMS preprint RIMS-851
- [6] G. Ponzano and T. Regge, in “Spectroscopic and Group Theoretical Methods in Physics” (North-Holland, Amsterdam, 1968)
- [7] E. Witten, Nucl. Phys. B311 (1988) 46
- [8] S. Carlip and S. P. De Alwis, Nucl. Phys. B337 (1990) 681
- [9] T. Kohno, Topology 31 No.2 (1992) 203
- [10] E. Witten, Commun. Math. Phys. 121 (1989) 351
- [11] S. Coleman, Nucl. Phys. B310 (1988) 643
- [12] E. Witten, Nucl. Phys. B323 (1989) 113
- [13] S. Carlip, UC Davis preprint UCD-92-16
- [14] S. Carlip, UC Davis preprint UCD-92-8
- [15] G. Moore and N. Seiberg, Commun. Math. Phys. 123 (1989) 177

- [16] M. Karowski, W. Muller and R. Schrader, Freie U. preprint (1991)
- [17] A. N. Kirillov and N. Y. Reshetikhin, in “New developments in the theory of knots” (World Scientific, 1989) 202
- [18] B. Hasslacher and M. J. Perry, Phys. Lett. 103B (1981) 21
- [19] K. Schulten and R. G. Gordon, J. Math. Phys. 16 (1961)1961;1975
- [20] K. Rama and S. Sen, Tata inst. preprint TIFR-TH-91-59
- [21] V. G. Turaev, C. R. Acad. Sci. Paris 313 I (1991) 395
- [22] E. Verlinde, Nucl. Phys. B300 (1988) 360
- [23] A. A. Belavin , A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
- [24] L. Alvarez-Gaume, C. Gomez and G. Sierra, Phys. Lett. B220 (1989) 142; Nucl. Phys. B330 (1990) 347
- [25] B. Durhuus, H. P. Jakobsen and R. Nest, Copenhagen preprint (1991)
- [26] A. Cappelli, C. Itzykson and J-B. Zuber, Nucl. Phys. B280 (1987) 445
- [27] R. Brustein, S. Yankielowicz and J-B. Zuber, Nucl. Phys. B313 (1989) 321
- [28] P. Bouwknegt and W. Nahm, Phys. Lett. B184 (1987) 359
- [29] P. Christe, Phys. Lett. 198B (1987) 215
- [30] J. Hempel, in “3-manifolds” (Princeton University Press, Princeton, 1976)

Figure 1: Subdivision dual to triangulation.

Figure 2: An Alexander move can be generated by compositions of (a)the bubble move \mathcal{B} , (b)the lune move \mathcal{L} and (c)the Matveev move \mathcal{M} .

Figure 3: A tetrahedron can be seen as a fusion matrix.

Figure 4: A convenient triangulation of a torus. Opposite sides of the rectangle are identified. Its dual graph represents a genus 3 conformal block.

Figure 5: The two genus 3 conformal blocks are related by fusing twice.

Figure 6: A triangulated cylinder $D^2 \times [0, 1]$.

Figure 7: $S^2 \times [0, 1]$ obtained by gluing two cylinders together.

Figure 8: B^3 obtained by gluing two tetrahedra.